

Charge Orbits and Moduli Spaces of Black Hole Attractors

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Abstract

We report on the theory of “large” U -duality charge orbits and related “moduli spaces” of extremal black hole attractors in $\mathcal{N} = 2$, $d = 4$ Maxwell-Einstein supergravity theories with symmetric scalar manifolds, as well as in $\mathcal{N} \geq 3$ -extended, $d = 4$ supergravities.

1 Introduction

The *Attractor Mechanism* (AM) [1] governs the dynamics in the scalar manifold of Maxwell-Einstein (super)gravity theories. It keeps standing as a crucial fascinating key topic within the international high-energy physics community. Along the last years, a number of papers have been devoted to the investigation of attractor configurations of extremal black p -branes in diverse space-time dimensions; for some lists of Refs., see *e.g.* [2].

The AM is related to dynamical systems with fixed points, describing the equilibrium state and the stability features of the system under consideration¹. When the AM holds, the particular property of the long-range behavior of the dynamical flows in the considered (dissipative) system is the following: in approaching the fixed points, properly named *attractors*, the orbits of the dynamical evolution lose all memory of their initial conditions, but however the overall dynamics remains completely deterministic.

The first example of AM in supersymmetric systems was discovered in the theory of static, spherically symmetric, asymptotically flat extremal dyonic black holes in $\mathcal{N}=2$ Maxwell-Einstein supergravity in $d=4$ and 5 space-time dimensions (see the first two Refs. of [1]). In the following, we will briefly present some basic facts about the $d=4$ case.

The multiplet content of a completely general $\mathcal{N}=2$, $d=4$ supergravity theory is the following (see *e.g.* [3], and Refs. therein):

1. the *gravitational* multiplet

$$(V_\mu^a, \psi^A, \psi_A, A^0), \quad (1.1)$$

described by the *Vielbein* one-form V^a ($a=0,1,2,3$) (together with the spin-connection one-form ω^{ab}), the $SU(2)$ doublet of gravitino one-forms ψ^A, ψ_A ($A=1,2$, with the upper and lower indices respectively denoting right and left chirality, *i.e.* $\gamma_5 \psi_A = -\gamma_5 \psi^A$), and the graviphoton one-form A^0 ;

2. n_V *vector* supermultiplets

$$(A^I, \lambda^{iA}, \bar{\lambda}_A^{\bar{i}}, z^i), \quad (1.2)$$

each containing a gauge boson one-form A^I ($I=1, \dots, n_V$), a doublet of gauginos (zero-form spinors) $\lambda^{iA}, \bar{\lambda}_A^{\bar{i}}$, and a complex scalar field (zero-form) z^i ($i=1, \dots, n_V$). The scalar fields z^i can be regarded as coordinates on a complex manifold \mathcal{M}_{n_V} ($\dim_{\mathbb{C}} \mathcal{M}_{n_V} = n_V$), which is actually a *special Kähler* manifold;

3. n_H *hypermultiplets*

$$(\zeta_\alpha, \zeta^\alpha, q^u), \quad (1.3)$$

each formed by a doublet of zero-form spinors, that is the hyperinos $\zeta_\alpha, \zeta^\alpha$ ($\alpha=1, \dots, 2n_H$), and four real scalar fields q^u ($u=1, \dots, 4n_H$), which can be considered as coordinates of a quaternionic manifold \mathcal{Q}_{n_H} ($\dim_{\mathbb{H}} \mathcal{Q}_{n_H} = n_H$).

At least in absence of gauging, the n_H hypermultiplets are spectators in the AM. This can be understood by looking at the transformation properties of the Fermi fields: the hyperinos $\zeta_\alpha, \zeta^\alpha$'s transform independently on the vector fields, whereas the gauginos' supersymmetry transformations depend on the Maxwell vector fields. Consequently, the contribution of the hypermultiplets can be dynamically decoupled from the rest of the physical system; in particular, it is also completely independent from the evolution dynamics of the complex scalars z^i 's coming from the vector multiplets (*i.e.* from the evolution flow in \mathcal{M}_{n_V}). By disregarding for simplicity's sake the fermionic and gauging terms, the supersymmetry transformations of hyperinos read (see *e.g.* [3], and Refs. therein)

$$\delta \zeta_\alpha = i \mathcal{U}_u^{B\beta} \partial_\mu q^u \gamma^\mu \varepsilon^A \epsilon_{AB} \mathbb{C}_{\alpha\beta}, \quad (1.4)$$

implying the asymptotical configurations of the quaternionic scalars of the hypermultiplets to be unconstrained, and therefore to vary continuously in the manifold \mathcal{Q}_{n_H} of the related quaternionic non-linear sigma model.

¹We recall that a point x_{fix} where the phase velocity $v(x_{fix})$ vanishes is called a *fixed* point, and it gives a representation of the considered dynamical system in its equilibrium state,

$$v(x_{fix}) = 0.$$

The fixed point is said to be an *attractor* of some motion $x(t)$ if

$$\lim_{t \rightarrow \infty} x(t) = x_{fix}.$$

Thus, as far as ungauged theories are concerned, for the treatment of AM one can restrict to consider $\mathcal{N} = 2$, $d = 4$ Maxwell-Einstein supergravity, in which n_V vector multiplets (1.2) are coupled to the gravity multiplet (1.1). The relevant dynamical system to be considered is the one related to the radial evolution of the configurations of complex scalar fields of such n_V vector multiplets. When approaching the event horizon of the black hole, the scalars dynamically run into fixed points, taking values which are only function (of the ratios) of the electric and magnetic charges associated to Abelian Maxwell vector potentials under consideration.

The inverse distance to the event horizon is the fundamental evolution parameter in the dynamics towards the fixed points represented by the *attractor* configurations of the scalar fields. Such near-horizon configurations, which “attracts” the dynamical evolutive flows in \mathcal{M}_{n_V} , are completely independent on the initial data of such an evolution, *i.e.* on the spatial asymptotical configurations of the scalars. Consequently, for what concerns the scalar dynamics, the system completely loses memory of its initial data, because the dynamical evolution is “attracted” by some fixed configuration points, purely depending on the electric and magnetic charges.

Recently, intriguing connections with the (quantum) theory of information arose out [4].

In the framework of supergravity theories, extremal black holes can be interpreted as BPS (Bogomol’ny-Prasad-Sommerfeld)-saturated [5] interpolating metric singularities in the low-energy effective limit of higher-dimensional superstrings or M -theory [6]. Their asymptotically relevant parameters include the ADM mass [7], the electrical and magnetic charges (defined by integrating the fluxes of related field strengths over the 2-sphere at infinity), and the asymptotical values of the (dynamically relevant set of) scalar fields. The AM implies that the class of black holes under consideration loses all its “scalar hair” within the near-horizon geometry. This means that the extremal black hole solutions, in the near-horizon limit in which they approach the Bertotti-Robinson $AdS_2 \times S^2$ conformally flat metric [8], are characterized only by electric and magnetic charges, but not by the continuously-varying asymptotical values of the scalar fields.

An important progress in the geometric interpretation of the AM was achieved in the last Ref. of [1], in which the attractor near-horizon scalar configurations were related to the critical points of a suitably defined black hole effective potential function V_{BH} , whose explicit form in maximal supergravity is *e.g.* given by Eq. (3.6) below. In general, V_{BH} is a positive definite function of scalar fields and electric and magnetic charges, and its non-degenerate critical points in \mathcal{M}_{n_V}

$$\forall i = 1, \dots, n_V, \frac{\partial V_{BH}}{\partial z^i} = 0 : \quad V_{BH}|_{\frac{\partial V_{BH}}{\partial z} = 0} > 0, \quad (1.5)$$

fix the scalar fields to depend only on electric and magnetic fluxes (charges). In the Einstein two-derivative approximation, the (semi)classical Bekenstein-Hawking entropy (S_{BH}) - area (A_H) formula [9] yields the (purely charge-dependent) black hole entropy S_{BH} to be

$$S_{BH} = \pi \frac{A_H}{4} = \pi V_{BH}|_{\frac{\partial V_{BH}}{\partial z} = 0} = \pi \sqrt{|\mathcal{I}_4|}, \quad (1.6)$$

where \mathcal{I}_4 is the unique independent invariant homogeneous polynomial (quartic in charges) in the relevant representation \mathbf{R}_V of G in which the charges sit (see Eq. (1.7) and discussion below). The last step of (1.6) does not apply to $d = 4$ supergravity theories with quadratic charge polynomial invariant, namely to the $\mathcal{N} = 2$ *minimally coupled* sequence [10] and to the $\mathcal{N} = 3$ [11] theory; in these cases, in (1.6) $\sqrt{|\mathcal{I}_4|}$ gets replaced by $|\mathcal{I}_2|$.

In presence of $n = n_V + 1$ Abelian vector fields, the fluxes sit in a $2n$ -dimensional representation \mathbf{R}_V of the U -duality group G , defining the embedding of G itself into $Sp(2n, \mathbb{R})$, which is the largest group acting linearly on the fluxes themselves:

$$G \stackrel{\mathbf{R}_V}{\hookrightarrow} Sp(2n, \mathbb{R}). \quad (1.7)$$

It should be pointed out that we here refer to U -duality as the continuous version of the U -duality groups introduced in [12]. This is consistent with the assumed (semi-)classical limit of large charges, also indicated by the fact that we consider $Sp(2n, \mathbb{R})$, and not $Sp(2n, \mathbb{Z})$ (no Dirac-Schwinger-Zwanziger quantization condition is implemented on the fluxes themselves).

After [13, 14, 15], the \mathbf{R}_V -representation space of the U -duality group is known to exhibit a stratification into disjoint classes of orbits, which can be defined through invariant sets of constraints on the (lowest order, actually unique) G -invariant \mathcal{I} built out of the symplectic representation \mathbf{R}_V . It is here worth remarking the crucial distinction between the “large” orbits and “small” orbits. While the former have $\mathcal{I} \neq 0$ and support an attractor behavior of the scalar flow in the near-horizon geometry of the extremal black hole background [1], for the latter the Attractor Mechanism does not hold, they have $\mathcal{I} = 0$ and thus they correspond to solutions with vanishing Bekenstein-Hawking [9] entropy (*at least* at the Einsteinian two-derivative level).

This short report, contributing to the Proceedings of the Workshop “*Supersymmetry in Mathematics and Physics*” (organized by Prof. R. Fiorese and Prof. V. S. Varadarajan), held on February 2010 at the Department of Mathematics of the University of California at Los Angeles, presents the main results of the theory of U -duality charge orbits and “moduli spaces” of extremal black hole attractor solutions in supergravity theories with $\mathcal{N} \geq 2$ supercharges in $d = 4$ space-time dimensions. In particular, $\mathcal{N} = 2$ Maxwell-Einstein theories with symmetric scalar manifolds will be considered.

The plan of this short review is as follows.

Sec. 2 introduces the “large” (*i.e.* attractor-supporting) charge orbits of the $\mathcal{N} = 2$, $d = 4$ *symmetric* Maxwell-Einstein supergravities, namely of those $\mathcal{N} = 2$ supergravity theories in which a certain number of Abelian vector multiplets is coupled to the gravity multiplet, and the corresponding complex scalars span a special Kähler manifold which is also a symmetric coset $\frac{G}{H_0 \times U(1)}$, where G is the U -duality group and $H_0 \times U(1)$ is its maximal compact subgroup.

Then, Sec. 3 is devoted to the analysis of the “large” charge orbits of the maximal $\mathcal{N} = 8$ supergravity theory. The non-compactness of the stabilizer groups of such (generally non-symmetric) coset orbits gives rise to the so-called “moduli spaces” of attractor solutions, namely proper subspaces of the scalar manifold of the theory in which the Attractor Mechanism is not active.

The “moduli spaces” of the various classes of non-supersymmetric attractors in $\mathcal{N} = 2$, $d = 4$ *symmetric* Maxwell-Einstein supergravities are then reported and discussed in Sec. 4.

The short Sec. 5 concludes the paper, analyzing the attractor-supporting orbits of $\mathcal{N} \geq 3$ -extended “pure” and matter-coupled theories, whose scalar manifolds are all symmetric.

2 Charge Orbits of $\mathcal{N} = 2$, $d = 4$ Symmetric Maxwell-Einstein Supergravities

$\mathcal{N} = 2$, $d = 4$ Maxwell-Einstein supergravity theories [16] with homogeneous symmetric special Kähler vector multiplets’ scalar manifolds $\frac{G}{H_0 \times U(1)}$ will be shortly referred to as *symmetric* Maxwell-Einstein supergravities. The various symmetric non-compact special Kähler spaces $\frac{G}{H_0 \times U(1)}$ (with $H_0 \times U(1)$ being the maximal compact subgroup with symmetric embedding (*mcs*) of G , the $d = 4$ U -duality group) have been classified in [17, 18] (see *e.g.* [19] for a recent account), and they are reported in Table 1.

All these theories can be obtained by dimensional reduction of the minimal $\mathcal{N} = 2$, $d = 5$ supergravities [16], and they all have cubic prepotential holomorphic functions. The unique exception is provided by the theories with \mathbb{CP}^n scalar manifolds, describing the *minimal coupling* of n Abelian vector multiplets to the gravity multiplet itself [10] (see also [20, 21]); in this case, the prepotential is quadratic in the scalar fields, and thus $C_{ijk} = 0$.

By disregarding the \mathbb{CP}^n sequence, the cubic prepotential of all these theories is related to the norm form of the Euclidean degree-3 Jordan algebra that defines them [16]. The reducible sequence in the third row of Table 1, usually referred to as the *generic Jordan family*, is based on the sequence of *reducible* Euclidean Jordan algebras $\mathbb{R} \oplus \mathbf{\Gamma}_{1,n-1}$, where \mathbb{R} denotes the 1-dimensional Jordan algebra and $\mathbf{\Gamma}_{1,n-1}$ stands for the degree-2 Jordan algebra with a quadratic form of Lorentzian signature $(1, n-1)$, which is nothing but the Clifford algebra of $O(1, n-1)$ [22].

Then, four other theories exist, defined by the irreducible degree-3 Jordan algebras $J_3^{\mathbb{O}}$, $J_3^{\mathbb{H}}$, $J_3^{\mathbb{C}}$ and $J_3^{\mathbb{R}}$, namely the Jordan algebras of Hermitian 3×3 matrices over the four division algebras \mathbb{O} (octonions), \mathbb{H} (quaternions), \mathbb{C} (complex numbers) and \mathbb{R} (real numbers) [16, 22, 23, 24, 25]. Because of their symmetry groups fit in the celebrated *Magic Square* of Freudenthal, Rozenfeld and Tits [26, 27], these theories have been named “*magic*”. By defining $A \equiv \dim_{\mathbb{R}} \mathbb{A}$ ($= 8, 4, 2, 1$ for $\mathbb{A} = \mathbb{O}, \mathbb{H}, \mathbb{C}, \mathbb{R}$, respectively), the complex dimension of the scalar manifolds of the “magic” Maxwell-Einstein theories is $3(A+1)$. It should also be recalled that the $\mathcal{N} = 2$ “magic” theory based on $J_3^{\mathbb{H}}$ shares the same bosonic sector with the $\mathcal{N} = 6$ “pure” supergravity (see *e.g.* [28, 29, 30]), and accordingly in this case the attractors enjoy a “dual” interpretation [20]. Furthermore, it should also be remarked that $J_2^{\mathbb{A}} \sim \mathbf{\Gamma}_{1,A+1}$ (see *e.g.* the eighth Ref. of [2]).

Within these theories, the “large” charge orbits, *i.e.* the ones supporting extremal black hole attractors have a non-maximal (nor generally symmetric) coset structure. The results [20] are reported in Table 2. After [13], the charge orbit supporting $(\frac{1}{2})$ -BPS attractors has coset structure

$$\mathcal{O}_{BPS} = \frac{G}{H_0}, \text{ with } H_0 \times U(1) \stackrel{mcs}{\subsetneq} G. \quad (2.1)$$

	$\frac{G}{H_0 \times U(1)}$	r	$\dim_{\mathbb{C}} \equiv n_V$
<i>minimal coupling</i> $n \in \mathbb{N}$	$\mathbb{CP}^n \equiv \frac{SU(1,n)}{U(1) \times SU(n)}$	1	n
$\mathbb{R} \oplus \mathbf{\Gamma}_{1,n-1}$, $n \in \mathbb{N}$	$\frac{SL(2,\mathbb{R})}{SO(2)} \times \frac{SO(2,n)}{SO(2) \times SO(n)}$	2 ($n=1$) 3 ($n \geq 2$)	$n+1$
$J_3^{\mathbb{O}}$	$\frac{E_{7(-25)}}{E_{6(-78)} \times U(1)}$	3	27
$J_3^{\mathbb{H}}$	$\frac{SO^*(12)}{U(6)}$	3	15
$J_3^{\mathbb{C}}$	$\frac{SU(3,3)}{S(U(3) \times U(3))}$	3	9
$J_3^{\mathbb{R}}$	$\frac{Sp(6,\mathbb{R})}{U(3)}$	3	6

Table 1: **Riemannian globally symmetric non-compact special Kähler spaces** (*alias* vector multiplets' scalar manifolds of the *symmetric* $\mathcal{N}=2$, $d=4$ Maxwell Einstein supergravity theories). r denotes the rank of the manifold, whereas n_V stands for the number of vector multiplets

	$\frac{1}{2}$ -BPS orbit $\mathcal{O}_{\frac{1}{2}\text{-BPS}} = \frac{G}{H_0}$	nBPS $Z_H \neq 0$ orbit $\mathcal{O}_{\text{nBPS}, Z_H \neq 0} = \frac{G}{H}$	nBPS $Z_H = 0$ orbit $\mathcal{O}_{\text{nBPS}, Z_H = 0} = \frac{G}{H}$
<i>minimal coupling</i> $n \in \mathbb{N}$	$\frac{SU(1,n)}{SU(n)}$	—	$\frac{SU(1,n)}{SU(1,n-1)}$
$\mathbb{R} \oplus \mathbf{\Gamma}_{1,n-1}$ $n \in \mathbb{N}$	$\frac{SL(2,\mathbb{R})}{SO(2)} \times \frac{SO(2,n)}{SO(n)}$	$\frac{SL(2,\mathbb{R})}{SO(1,1)} \times \frac{SO(2,n)}{SO(1,n-1)}$	$\frac{SL(2,\mathbb{R})}{SO(2)} \times \frac{SO(2,n)}{SO(2,n-2)}$
$J_3^{\mathbb{O}}$	$\frac{E_{7(-25)}}{E_6}$	$\frac{E_{7(-25)}}{E_{6(-26)}}$	$\frac{E_{7(-25)}}{E_{6(-14)}}$
$J_3^{\mathbb{H}}$	$\frac{SO^*(12)}{SU(6)}$	$\frac{SO^*(12)}{SU^*(6)}$	$\frac{SO^*(12)}{SU(4,2)}$
$J_3^{\mathbb{C}}$	$\frac{SU(3,3)}{SU(3) \times SU(3)}$	$\frac{SU(3,3)}{SL(3,\mathbb{C})}$	$\frac{SU(3,3)}{SU(2,1) \times SU(1,2)}$
$J_3^{\mathbb{R}}$	$\frac{Sp(6,\mathbb{R})}{SU(3)}$	$\frac{Sp(6,\mathbb{R})}{SL(3,\mathbb{R})}$	$\frac{Sp(6,\mathbb{R})}{SU(2,1)}$

Table 2: **Charge orbits of attractors in *symmetric* $\mathcal{N}=2$, $d=4$ Maxwell-Einstein supergravities**

As shown in [20], there are other two charge orbits supporting extremal black hole attractors, and they are both non-supersymmetric (not saturating the BPS bound [5]). One has non-vanishing $\mathcal{N}=2$ central charge at the

horizon ($Z_H \neq 0$), with coset structure

$$\mathcal{O}_{nBPS, Z_H \neq 0} = \frac{G}{\widehat{H}}, \text{ with } \widehat{H} \times SO(1, 1) \subsetneq G, \quad (2.2)$$

where \widehat{H} denotes the $d = 5$ U -duality group, and thus $SO(1, 1)$ corresponds to the S^1 -radius in the Kaluza-Klein reduction $d = 5 \rightarrow 4$. Also the remaining attractor-supporting charge orbit is non-supersymmetric, but it corresponds to $Z_H = 0$; its coset structure reads

$$\mathcal{O}_{nBPS, Z_H = 0} = \frac{G}{\widetilde{H}}, \text{ with } \widetilde{H} \times U(1) \subsetneq G. \quad (2.3)$$

It is worth remarking that \widehat{H} and \widetilde{H} are the only two non-compact forms of H_0 such that the group embedding in the right-hand side of (2.3) and (2.2) are both maximal and symmetric (see *e.g.* [31, 32, 33]).

Due to (2.1), H_0 is the maximal compact symmetry group of the particular class of non-degenerate critical points of the effective black hole potential V_{BH} corresponding to BPS attractors. On the other hand, the maximal compact symmetry group of the non-BPS $Z_H \neq 0$ and non-BPS $Z_H = 0$ critical points of V_{BH} respectively is

$$\widehat{h} = mcs(\widehat{H}); \quad \widetilde{h} = mcs(\widetilde{H}). \quad (2.4)$$

Actually, in the non-BPS $Z_H = 0$ case, the maximal compact symmetry is $\widetilde{h}' \equiv \frac{\widetilde{h}}{U(1)}$; see *e.g.* [20] for further details.

General results on the rank \mathfrak{r} of the $2n_V \times 2n_V$ Hessian matrix \mathbf{H} of V_{BH} are known. Firstly, the BPS (non-degenerate) critical points of $V_{BH, \mathcal{N}=2}$ are stable, and thus \mathbf{H}_{BPS} has no massless modes (see the fifth Ref. of [1]), and its rank is maximal: $\mathfrak{r}_{BPS} = 2n_V$. Furthermore, the analysis of [20] showed that for the other two classes of (non-degenerate) non-supersymmetric critical points of $V_{BH, \mathcal{N}=2}$, the rank of \mathbf{H} is model-dependent:

$$\mathbb{CP}^n : \quad \mathfrak{r}_{nBPS, Z_H=0} = 2; \quad (2.5)$$

$$\mathbb{R} \oplus \mathbf{\Gamma}_{1, n-1} : \quad \begin{cases} \mathfrak{r}_{nBPS, Z_H \neq 0} = n + 2; \\ \mathfrak{r}_{nBPS, Z_H=0} = 6; \end{cases} \quad (2.6)$$

$$J_3^{\mathbb{A}} : \quad \begin{cases} \mathfrak{r}_{nBPS, Z_H \neq 0} = 3A + 4; \\ \mathfrak{r}_{nBPS, Z_H=0} = 2A + 6. \end{cases} \quad (2.7)$$

3 $\mathcal{N} = 8$, $d = 4$ Supergravity

The analysis of extremal black hole attractors in the theory with the maximal number of supercharges, namely in $\mathcal{N} = 8$, $d = 4$ supergravity, provides a simpler, warm-up framework for the analysis and classification of the “moduli spaces” of the two classes ($Z_H \neq 0$ and $Z_H = 0$) of non-BPS attractors of quarter-minimal Maxwell-Einstein supergravities with symmetric scalar manifolds, which have been introduced in Sec. 2.

Maximal supergravity in four dimensions is based on the real, rank-7, 70-dimensional homogeneous symmetric manifold

$$\frac{G_{\mathcal{N}=8}}{H_{\mathcal{N}=8}} = \frac{E_{7(7)}}{SU(8)}, \quad (3.1)$$

where $SU(8) = mcs(E_{7(7)})$. After [13, 14, 15, 34, 35], two classes of (non-degenerate) critical points of $V_{BH, \mathcal{N}=8}$ are known to exist:

- the $\frac{1}{8}$ -BPS class, supported by the orbit

$$\mathcal{O}_{\frac{1}{8}-BPS, \mathcal{N}=8} \equiv \frac{G_{\mathcal{N}=8}}{\mathcal{H}_{\mathcal{N}=8}} = \frac{E_{7(7)}}{E_{6(2)}}, \quad E_{6(2)} \times U(1) \subsetneq E_{7(7)}; \quad (3.2)$$

- the non-BPS class, supported by the orbit

$$\mathcal{O}_{nBPS, \mathcal{N}=8} \equiv \frac{G_{\mathcal{N}=8}}{\widehat{\mathcal{H}}_{\mathcal{N}=8}} = \frac{E_{7(7)}}{E_{6(6)}}, \quad E_{6(6)} \times SO(1, 1) \subsetneq E_{7(7)}. \quad (3.3)$$

Both charge orbits $\mathcal{O}_{\frac{1}{8}\text{-BPS}, \mathcal{N}=8}$ and $\mathcal{O}_{n\text{BPS}, \mathcal{N}=8}$ belong to the fundamental representation space **56** of the maximally non-compact (split) form $E_{7(7)}$ of the exceptional group E_7 . The embeddings in the right-hand side of Eqs. (3.2) and (3.3) are both maximal and symmetric (see *e.g.* [31, 33]). Among all non-compact forms of the exceptional Lie group E_6 (*i.e.* $E_{6(-26)}$, $E_{6(-14)}$, $E_{6(2)}$ and $E_{6(6)}$), $E_{6(2)}$ and $E_{6(6)}$ are the only two which are maximally and symmetrically embedded (through an extra group factor $U(1)$ or $SO(1,1)$) into $E_{7(7)}$.

In the maximal theory, the Hessian matrix $\mathbf{H}_{\mathcal{N}=8}$ of the effective potential $V_{BH, \mathcal{N}=8}$ is a square 70×70 symmetric matrix. At $\frac{1}{8}$ -BPS attractor points, $\mathbf{H}_{\mathcal{N}=8}$ has rank 30, with 40 massless modes [36] sitting in the representation **(20, 2)** of the enhanced $\frac{1}{8}$ -BPS symmetry group $SU(6) \times SU(2) = mcs(\mathcal{H}_{\mathcal{N}=8})$ [35]. Moreover, at non-BPS attractor points, $\mathbf{H}_{\mathcal{N}=8}$ has rank 28, with 42 massless modes sitting in the representation **42** of the enhanced non-BPS symmetry group $USp(8) = mcs(\hat{\mathcal{H}}_{\mathcal{N}=8})$ [35]. Actually, the massless modes of $\mathbf{H}_{\mathcal{N}=8}$ are “flat” directions of $V_{BH, \mathcal{N}=8}$ at the corresponding classes of its critical points. Thus, such “flat” directions of the critical $V_{BH, \mathcal{N}=8}$ span some “moduli spaces” of the attractor solutions [37], corresponding to the scalar degrees of freedom which are not stabilized by the *Attractor Mechanism* [1] at the black hole event horizon. In the $\mathcal{N} = 8$ case, such “moduli spaces” are the following real symmetric sub-manifolds of $\frac{E_{7(7)}}{SU(8)}$ itself [37]:

$$\frac{1}{8}\text{-BPS} \quad : \quad \mathcal{M}_{\frac{1}{8}\text{-BPS}} = \frac{\mathcal{H}_{\mathcal{N}=8}}{mcs(\mathcal{H}_{\mathcal{N}=8})} = \frac{E_{6(2)}}{SU(6) \times SU(2)}, \quad \dim_{\mathbb{R}} = 40, \text{ rank} = 4; \quad (3.4)$$

$$\text{non-BPS} \quad : \quad \mathcal{M}_{n\text{BPS}} = \frac{\hat{\mathcal{H}}_{\mathcal{N}=8}}{mcs(\hat{\mathcal{H}}_{\mathcal{N}=8})} = \frac{E_{6(6)}}{USp(8)}, \quad \dim_{\mathbb{R}} = 42, \text{ rank} = 6. \quad (3.5)$$

It is easy to realize that $\mathcal{M}_{\frac{1}{8}\text{-BPS}}$ and $\mathcal{M}_{n\text{BPS}}$ are nothing but the cosets of the non-compact stabilizer of the corresponding supporting charge orbit ($E_{6(2)}$ and $E_{6(6)}$, respectively) and of its *mcs*. Actually, this is the very structure of all “moduli spaces” of attractors (see Sects. 4 and 5). Moreover, $\mathcal{M}_{n\text{BPS}}$ is nothing but the scalar manifold of $\mathcal{N} = 8$, $d = 5$ supergravity. This holds more in general, and, as given by the treatment of Sec. 4 (see also Table 3), the “moduli space” of $\mathcal{N} = 2$, $d = 4$ non-BPS $Z_H \neq 0$ attractors is nothing but the scalar manifold of the $d = 5$ uplift of the corresponding theory [37] (see also [38]).

Following [37] and considering the maximal supergravity theory, we now explain the reason why the “flat” directions of the Hessian matrix of the effective potential at its critical points actually span a “moduli space” (for a recent discussion, see also [39]).

Let us start by recalling that $V_{BH, \mathcal{N}=8}$ is defined as

$$V_{BH, \mathcal{N}=8} \equiv \frac{1}{2} Z_{AB}(\phi, Q) \bar{Z}^{AB}(\phi, Q), \quad (3.6)$$

where Z_{AB} is the antisymmetric complex $\mathcal{N} = 8$ central charge matrix [14]

$$Z_{AB}(\phi, Q) = (Q^T L(\phi))_{AB} = (Q^T)_{\Lambda} L_{AB}^{\Lambda}(\phi). \quad (3.7)$$

ϕ denotes the 70 real scalar fields parametrising the aforementioned coset $\frac{E_{7(7)}}{SU(8)}$, Q is the $\mathcal{N} = 8$ charge vector sitting in the fundamental irrepr. **56** of the U -duality group $E_{7(7)}$. Moreover, $L_{AB}^{\Lambda}(\phi)$ is the ϕ -dependent coset representative, *i.e.* a local section of the principal bundle $E_{7(7)}$ over $\frac{E_{7(7)}}{SU(8)}$ with structure group $SU(8)$.

The action of an element $g \in E_{7(7)}$ on $V_{BH, \mathcal{N}=8}(\phi, Q)$ is such that

$$V_{BH, \mathcal{N}=8}(\phi, Q) = V_{BH, \mathcal{N}=8}(\phi_g, Q^g) = V_{BH, \mathcal{N}=8}(\phi_g, (g^{-1})^T Q); \quad (3.8)$$

thus, $V_{BH, \mathcal{N}=8}$ is not $E_{7(7)}$ -invariant, because its coefficients (given by the components of Q) do not in general remain the same. The situation changes if one restricts $g \equiv g_Q \in H_Q$ to belong to the stabilizer H_Q of one of the orbits $\frac{E_{7(7)}}{H_Q}$ spanned by the charge vector Q within the **56** representation space of $E_{7(7)}$ itself. In such a case:

$$Q^{g_Q} = Q \Rightarrow V_{BH, \mathcal{N}=8}(\phi, Q) = V_{BH, \mathcal{N}=8}(\phi_{g_Q}, Q). \quad (3.9)$$

Then, it is natural to split the 70 real scalar fields ϕ as $\phi = \{\phi_Q, \check{\phi}_Q\}$, where $\phi_Q \in \frac{H_Q}{mcs(H_Q)} \subsetneq \frac{E_{7(7)}}{SU(8)}$ and $\check{\phi}_Q$ coordinatise the complement of $\frac{H_Q}{mcs(H_Q)}$ in $\frac{E_{7(7)}}{SU(8)}$. By denoting with

$$V_{BH, \mathcal{N}=8, \text{crit}}(\phi_Q, Q) \equiv V_{BH, \mathcal{N}=8}(\phi, Q)|_{\frac{\partial V_{BH, \mathcal{N}=8}}{\partial \check{\phi}_Q} = 0} (\neq 0) \quad (3.10)$$

the values of $V_{BH,\mathcal{N}=8}$ along the equations of motion for the scalars $\check{\phi}_Q$, the invariance of $V_{BH,\mathcal{N}=8,crit}(\phi_Q, Q)$ under H_Q directly follows from Eq. (3.9) :

$$V_{BH,\mathcal{N}=8,crit}\left((\phi_Q)_{g_Q}, Q\right) = V_{BH,\mathcal{N}=8,crit}(\phi_Q, Q). \quad (3.11)$$

Now, it is crucial to observe that H_Q generally is a *non-compact* Lie group; for instance, $H_Q = E_{6(2)} \equiv \mathcal{H}_{\mathcal{N}=8}$ for $Q \in \mathcal{O}_{\frac{1}{8}-BPS,\mathcal{N}=8}$ given by (3.2), and $H_Q = E_{6(6)} \equiv \hat{\mathcal{H}}_{\mathcal{N}=8}$ for $Q \in \mathcal{O}_{nBPS,\mathcal{N}=8}$ given by (3.3). This implies $V_{BH,\mathcal{N}=8}$ to be independent *at its critical points* on the subset

$$\phi_Q \in \frac{H_Q}{\text{mcs}(H_Q)} \subsetneq \frac{E_{7(7)}}{SU(8)}. \quad (3.12)$$

Thus, $\frac{H_Q}{\text{mcs}(H_Q)}$ can be regarded as the “moduli space” of the attractor solutions supported by the charge orbit $\frac{E_{7(7)}}{H_Q}$. For $\mathcal{N} = 8$ non-degenerate critical points, supported by $\mathcal{O}_{\frac{1}{8}-BPS,\mathcal{N}=8}$ and $\mathcal{O}_{nBPS,\mathcal{N}=8}$, this reasoning yields to the “moduli spaces” $\mathcal{M}_{\frac{1}{8}-BPS}$ and \mathcal{M}_{nBPS} , respectively given by (3.4) and (3.3).

The results on $\mathcal{N} = 8$ theory are summarized in the last row of Tables 5 and 6.

The above arguments apply to a general, not necessarily supersymmetric, Maxwell-Einstein theory with scalars coordinatising an homogeneous (not necessarily symmetric) space. In particular, one can repeat the above reasoning for all supergravities with $\mathcal{N} \geq 1$ based on homogeneous (not necessarily symmetric) manifolds $\frac{G_{\mathcal{N}}}{\text{mcs}(G_{\mathcal{N}})} \equiv \frac{G_{\mathcal{N}}}{\text{mcs}(G_{\mathcal{N}})}$, also in presence of matter multiplets. It is here worth recalling that theories with $\mathcal{N} \geq 3$ all have symmetric scalar manifolds (see *e.g.* [28]).

A remarkable consequence is the existence of “moduli spaces” of attractors is the following. By choosing Q belonging to the orbit $\frac{G_{\mathcal{N}}}{H_Q} \subsetneq \mathbf{R}_V(G_{\mathcal{N}})$ and supporting a class of non-degenerate critical points of $V_{BH,\mathcal{N}}$, *up to some “flat” directions* (spanning the “moduli space” $\frac{H_Q}{\text{mcs}(H_Q)} \subsetneq \frac{G_{\mathcal{N}}}{H_{\mathcal{N}}}$), *all* such critical points of $V_{BH,\mathcal{N}}$ in *all* $\mathcal{N} \geq 0$ Maxwell-Einstein (super)gravities with an homogeneous (not necessarily symmetric) scalar manifold (also in presence of matter multiplets) are *stable*, and thus they are *attractors* in a generalized sense. For $d = 4$ supergravities, $H_Q = \mathcal{H}$, $\hat{\mathcal{H}}$ or $\tilde{\mathcal{H}}$ (see *e.g.* Tables 5 and 6; see the third, fifth and seventh Refs. of [2]).

All this reasoning can be extended to a number of space-time dimensions $d \neq 4$ (see *e.g.* [40, 41, 42, 43]). As found in [44, 45] for “large” charge orbits of $\mathcal{N} = 2$, $d = 4$ *stu* model, and then proved in a model-independent way in [39], the “moduli spaces” of charge orbits are defined *all along the corresponding scalar flows*, and thus they can be interpreted as “moduli spaces” of unstabilized scalars at the event horizon of the extremal black hole, as well as “moduli spaces” of the ADM mass [7] of the extremal black hole at spatial infinity.

Remarkably, one can associate “moduli spaces” also to non-attractive, “small” orbits, namely to charge orbits supporting black hole configurations which have vanishing horizon area in the Einsteinian approximation [46, 47, 43]. Differently from “large” orbits, for “small” orbits there exists a “moduli space” also when the semi-simple part of H_Q is compact, and it has translational nature [43]. Clearly, in the “small” case the interpretation at the event horizon breaks down, simply because such an horizon does not exist at all, *at least* in Einsteinian supergravity approximation.

4 “Moduli Spaces” of Attractors in $\mathcal{N} = 2$, $d = 4$ Symmetric Maxwell-Einstein Supergravities

The arguments outlined in Sec. 3 can be used to determine the “moduli spaces” of non-BPS attractors (with $Z_H \neq 0$ or $Z_H = 0$) for all $\mathcal{N} = 2$, $d = 4$ Maxwell-Einstein supergravities with symmetric scalar manifolds [37].

After the fifth Ref. of [1], it is known that, regardless of the geometry of the vector multiplets’ scalar manifold, the BPS non-degenerate critical points of $V_{BH,\mathcal{N}=2}$ are *stable*, and thus define an attractor configuration in strict sense, in which all scalar fields are stabilized in terms of charges by the Attractor Mechanism [1]. This is ultimately due to the fact that the Hessian matrix $\mathbf{H}_{\frac{1}{2}-BPS}$ at such critical points has no massless modes at all. Therefore, as far as the metric of the scalar manifold is non-singular and positive-definite and no massless degrees of freedom appear in the theory, there is no “moduli space” for BPS attractors in *any* $\mathcal{N} = 2$, $d = 4$ Maxwell-Einstein supergravity theory.

This is an important difference with respect to $\frac{1}{\mathcal{N}}$ -BPS attractors in $\mathcal{N} > 2$ -extended supergravities (see the third, fifth and seventh Refs. of [2]; for instance, in $\mathcal{N} = 8$ theory $\frac{1}{8}$ -BPS attractors exhibit the “moduli space” $\mathcal{M}_{\frac{1}{8}-BPS}$ given by (3.4). From a group theoretical perspective, such a difference can be ascribed to

the *compactness* of the stabilizer H_0 of the “large” BPS charge orbit $\mathcal{O}_{\frac{1}{2}\text{-BPS}, \mathcal{N}=2}$ in the $\mathcal{N} = 2$ symmetric case (see Table 3). From a supersymmetry perspective, such a difference can be traced back to the different degrees of supersymmetry preservation exhibited by attractor solutions in theories with a different number \mathcal{N} of supercharges. Indeed, $(\frac{1}{2}\text{-})$ BPS attractors in theories with local $\mathcal{N} = 2$ supersymmetry are *maximally* supersymmetric (namely, they preserve the *maximum* number of supersymmetries out of the ones related to the asymptotical Poincaré background). On the other hand, in \mathcal{N} -extended ($2 < \mathcal{N} \leq 8$) supergravities BPS attractors correspond to $\frac{1}{\mathcal{N}}$ -BPS configurations, which are *not maximally* supersymmetric. In these latter theories, the maximally supersymmetric configurations correspond to vanishing black hole entropy (at the two-derivative Einsteinian level).

Exploiting the observation below Eq. (3.3), it is possible to determine the “moduli spaces” of non-BPS critical points ($Z_H \neq 0$ or $Z_H = 0$) of $V_{BH, \mathcal{N}=2}$ for all $\mathcal{N} = 2$, $d = 4$ Maxwell-Einstein supergravities with symmetric scalar manifold. Consistent with the notation introduced in Sec. 2 (recall (2.4)), the $\mathcal{N} = 2$ non-BPS $Z_H \neq 0$ and $Z_H = 0$ “moduli spaces” are respectively denoted by (see [20, 37] for further details on notation)

$$\mathcal{M}_{nBPS, Z_H \neq 0} = \frac{\hat{H}}{\text{mcs}(\hat{H})} \equiv \frac{\hat{H}}{h}; \quad (4.1)$$

$$\mathcal{M}_{nBPS, Z_H = 0} = \frac{\tilde{H}}{\text{mcs}(\tilde{H})} \equiv \frac{\tilde{H}}{h} = \frac{\tilde{H}}{h' \times U(1)}. \quad (4.2)$$

The results are reported in Tables 3 and 4 [37].

	$\frac{\hat{H}}{\text{mcs}(\hat{H})}$	r	$\dim_{\mathbb{R}}$
$\mathbb{R} \oplus \mathbf{\Gamma}_{1, n-1}, \quad n \in \mathbb{N}$	$SO(1, 1) \times \frac{SO(1, n-1)}{SO(n-1)}$	1 ($n = 1$) 2 ($n \geq 2$)	n
$J_3^{\mathbb{O}}$	$\frac{E_{6(-26)}}{F_{4(-52)}}$	2	26
$J_3^{\mathbb{H}}$	$\frac{SU^*(6)}{USp(6)}$	2	14
$J_3^{\mathbb{C}}$	$\frac{SL(3, \mathbb{C})}{SU(3)}$	2	8
$J_3^{\mathbb{R}}$	$\frac{SL(3, \mathbb{R})}{SO(3)}$	2	5

Table 3: “Moduli spaces” of non-BPS $Z_H \neq 0$ critical points of $V_{BH, \mathcal{N}=2}$ in $\mathcal{N} = 2$, $d = 4$ *symmetric* Maxwell-Einstein supergravities. They are the $\mathcal{N} = 2$, $d = 5$ symmetric real special manifolds

As observed below Eq. (3.5), the non-BPS $Z_H \neq 0$ “moduli spaces” are nothing but the scalar manifolds of minimal ($\mathcal{N} = 2$) Maxwell-Einstein supergravity in $d = 5$ space-time dimensions [16]. Their real dimension $\dim_{\mathbb{R}}$ (rank r) is the complex dimension $\dim_{\mathbb{C}}$ (rank r) of the $\mathcal{N} = 2$, $d = 4$ symmetric special Kähler manifolds listed in Table 1, minus one. With the exception of the $n = 1$ element of the generic Jordan family $\mathbb{R} \oplus \mathbf{\Gamma}_{1, n-1}$ (the so-called st^2 model) having $\frac{\hat{H}}{h} = SO(1, 1)$ with rank $r = 1$, all non-BPS $Z_H \neq 0$ “moduli spaces” have rank $r = 2$. The results reported in Table 3 are consistent with the “ $n_V + 1 / n_V - 1$ ” mass degeneracy splitting of non-BPS $Z_H \neq 0$ attractors [48, 20, 49, 35], holding for a generic special Kähler cubic geometry of complex dimension n_V .

The non-BPS $Z_H = 0$ “moduli spaces”, reported in Table 4, are symmetric (generally non-special) Kähler manifolds. Note that in the $n = 1$ and $n = 2$ elements of the generic Jordan family $\mathbb{R} \oplus \mathbf{\Gamma}_{1, n-1}$ (the so-called st^2 and stu models, respectively), there are no non-BPS $Z_H = 0$ “flat” directions at all (see Appendix II of [20],

	$\frac{\tilde{H}}{mcs(H)} \equiv \frac{\tilde{H}}{h' \times U(1)}$	r	$\dim_{\mathbb{C}}$
<i>minimal coupling</i> $n \in \mathbb{N}$	$\frac{SU(1,n-1)}{U(1) \times SU(n-1)}$	1	$n - 1$
$\mathbb{R} \oplus \Gamma_{1,n-1}, n \in \mathbb{N}$	$\frac{SO(2,n-2)}{SO(2) \times SO(n-2)}, n \geq 3$	1 ($n = 3$) 2 ($n \geq 4$)	$n - 2$
$J_3^{\mathbb{O}}$	$\frac{E_{6(-14)}}{SO(10) \times U(1)}$	2	16
$J_3^{\mathbb{H}}$	$\frac{SU(4,2)}{SU(4) \times SU(2) \times U(1)}$	2	8
$J_3^{\mathbb{C}}$	$\frac{SU(2,1)}{SU(2) \times U(1)} \times \frac{SU(1,2)}{SU(2) \times U(1)}$	2	4
$J_3^{\mathbb{R}}$	$\frac{SU(2,1)}{SU(2) \times U(1)}$	1	2

Table 4: “Moduli spaces” of non-BPS $Z_H = 0$ critical points of $V_{BH, \mathcal{N}=2}$ in $\mathcal{N} = 2, d = 4$ *symmetric* Maxwell-Einstein supergravities. They are (non-special) symmetric Kähler manifolds

and [37]). By recalling the definition $A \equiv \dim_{\mathbb{R}} \mathbb{A}$ given above, the results reported in Table 4 [37] imply that the the non-BPS $Z_H = 0$ “moduli spaces” of $\mathcal{N} = 2, d = 4$ “magic” supergravities have complex dimension $2A$. As observed in [37], the non-BPS $Z_H = 0$ “moduli space” of $\mathcal{N} = 2, d = 4$ “magic” supergravity associated to $J_3^{\mathbb{O}}$ is the manifold $\frac{E_{6(-14)}}{SO(10) \otimes U(1)}$, which is related to another exceptional Jordan triple system over \mathbb{O} , as found long time ago in [16].

5 $\mathcal{N} \geq 3$ -Extended, $d = 4$ Supergravities

As anticipated above, the scalar manifolds of all $d = 4$ supergravity theories with $\mathcal{N} \geq 3$ supercharges are symmetric spaces (they are reported *e.g.* in Table 6 of [19]). Both $\frac{1}{\mathcal{N}}$ -BPS and non-BPS attractors exhibit a related “moduli space”. An example is provided by the maximal theory, already reviewed in Sec. 3. As mentioned above, the *non-compactness* of the stabilizer group of the corresponding supporting charge orbit is the ultimate reason of the existence of the “moduli spaces” of attractor solutions [35, 37] (see also the fifth Ref. of [2]).

By performing a supersymmetry truncation down to $\mathcal{N} = 2$ [50, 36, 35], the $\frac{1}{\mathcal{N}}$ -BPS “flat” directions of $V_{BH, \mathcal{N}}$ can be interpreted in terms of left-over $\mathcal{N} = 2$ hypermultiplets’ scalar degrees of freedom. As studied in [35], for non-BPS “flat” directions the situation is more involved, and an easy interpretation in terms of truncated-away hypermultiplets’ scalars degrees of freedom is generally lost.

Tables 5 and 6 report all classes of charge orbits supporting attractor solutions in $\mathcal{N} \geq 3$ -extended supergravity theories in $d = 4$ space-time dimensions (see the third, fifth and seventh Refs. of [2]).

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	$\frac{1}{\mathcal{N}}$ -BPS orb $\frac{G_{\mathcal{N}}}{\mathcal{H}_{\mathcal{N}}}$	nBPS $Z_{AB,H} \neq 0$ orb $\frac{G_{\mathcal{N}}}{\mathcal{H}_{\mathcal{N}}}$	nBPS $Z_{AB,H} = 0$ orb $\frac{G_{\mathcal{N}}}{\mathcal{H}_{\mathcal{N}}}$
$\mathcal{N} = 3$ $n \in \mathbb{N}$	$\frac{SU(3,n)}{SU(2,n)}$	—	$\frac{SU(3,n)}{SU(3,n-1)}$
$\mathcal{N} = 4$ $n \in \mathbb{N}, \mathbb{R} \oplus \mathbf{\Gamma}_{5,n-1}$	$\frac{SL(2,\mathbb{R})}{SO(2)} \times \frac{SO(6,n)}{SO(4,n)}$	$\frac{SL(2,\mathbb{R})}{SO(1,1)} \times \frac{SO(6,n)}{SO(5,n-1)}$	$\frac{SL(2,\mathbb{R})}{SO(2)} \times \frac{SO(6,n)}{SO(6,n-2)}$
$\mathcal{N} = 5$ $M_{1,2}(\mathbb{O})$	$\frac{SU(1,5)}{SU(3) \times SU(2,1)}$	—	—
$\mathcal{N} = 6$ $J_3^{\mathbb{H}}$	$\frac{SO^*(12)}{SU(4,2)}$	$\frac{SO^*(12)}{SU^*(6)}$	$\frac{SO^*(12)}{SU(6)}$
$\mathcal{N} = 8$ $J_3^{\mathbb{O}_s}$	$\frac{E_{7(7)}}{E_{6(2)}}$	$\frac{E_{7(7)}}{E_{6(6)}}$	—

Table 5: **Charge orbits supporting extremal black hole attractors in $\mathcal{N} \geq 3$ -extended, $d = 4$ supergravities** (n is the number of matter multiplets) (see the fifth Ref. of [2]). The related Euclidean degree-3 Jordan algebra is also given (*if any*). $M_{1,2}(\mathbb{O})$ is the Jordan triple system (not upliftable to $d = 5$) generated by 2×1 Hermitian matrices over \mathbb{O} [16].

	$\frac{1}{\mathcal{N}}$ -BPS “moduli space” $\frac{\mathcal{H}_{\mathcal{N}}}{mcs(\mathcal{H}_{\mathcal{N}})}$	nBPS $Z_{AB,H} \neq 0$ “moduli space” $\frac{\hat{\mathcal{H}}_{\mathcal{N}}}{mcs(\mathcal{H}_{\mathcal{N}})}$	nBPS $Z_{AB,H} = 0$ “moduli space” $\frac{\tilde{\mathcal{H}}_{\mathcal{N}}}{mcs(\mathcal{H}_{\mathcal{N}})}$
$\mathcal{N} = 3$	$\frac{SU(2,n)}{SU(2) \times SU(n) \times U(1)}$	—	$\frac{SU(3,n-1)}{SU(3) \times SU(n-1) \times U(1)}$
$\mathcal{N} = 4$	$\frac{SO(4,n)}{SO(4) \times SO(n)}$	$SO(1,1) \times \frac{SO(5,n-1)}{SO(5) \times SO(n-1)}$	$\frac{SO(6,n-2)}{SO(6) \times SO(n-2)}$
$\mathcal{N} = 5$	$\frac{SU(2,1)}{SU(2) \times U(1)}$	—	—
$\mathcal{N} = 6$	$\frac{SU(4,2)}{SU(4) \times SU(2) \times U(1)}$	$\frac{SU^*(6)}{USp(6)}$	—
$\mathcal{N} = 8$	$\frac{E_{6(2)}}{SU(6) \times SU(2)}$	$\frac{E_{6(6)}}{USp(8)}$	—

Table 6: **“Moduli spaces” of black hole attractor solutions in $\mathcal{N} \geq 3$ -extended, $d = 4$ supergravities.** n is the number of matter multiplets (see the fifth Ref. of [2])

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